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Null and infinitesimal isotropy in semi-Riemannian geometry

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Null and infinitesimal isotropy are defined for semi-Riemannian manifolds in a more general context. A theorem of Karcher is extended to semi-Riemannian manifolds in a more general setting. Also, by using this theorem, a characterization of static blackhole metrics can be made as well as a characterization of Robertson–Walker metrics as Karcher made.

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1. Introduction

A characterization of Robertson–Walker metrics was first made by Karcher [8] by introducing the concept of infinitesimal spatial isotropy. He also obtained a local decomposition for such manifolds. Later Harris [5] and Koch-Sen [9] obtained an equivalent characterization of those metrics by introducing the concept of infinitesimal null isotropy. Nevertheless, these characterizations seem to be somewhat restrictive for cosmological circumstances. Hence we shall introduce the concept of null isotropy in the first place, which is weaker than infinitesimal null isotropy and may be expected to be satisfied in cosmological circumstances. An important feature of this definition is that the curvature tensor can be determined by the Ricci tensor, which is determined by the stress–energy ten-

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sor via the Einstein equation. We shall also obtain a local decomposition theorem for such spacetimes.

Actually, the concept of infinitesimal spatial isotropy was given to fit Robertson–Walker spacetimes. However, an extension of this concept also fits static blackhole metrics. Hence we will also introduce an extended concept of infinitesimal isotropy and obtain generalizations of Karcher's decomposition theorem which also fit static black hole circumstances in characterizing these metrics.

Actually the methods we shall use are more general than to hold only in Lorentzian manifolds. Therefore we shall state these theorems in full generality, that is, for semi-Riemannian manifolds. The interested reader may also consider these theorems in Lorentzian Geometry or General Relativity.

2. Preliminaries

Let M denote an n-dimensional semi-Riemannian manifold with metric $\langle\ ,\ \rangle$ of index ν, where $0\leqslant\nu\leqslant n$. For a null $u\in T_pM$, define $\bar{u}^\perp=u^\perp/\mathrm{span}\{u\}$, where $u^\perp$ is the orthogonal space to $\mathrm{span}\{u\}$, and define the induced metric on $\bar{u}^\perp$ by $\bar{g}(\bar{x},\bar{y})=g(x,y)$, where $x, y\in u^\perp$ with $\pi(x)=\bar{x}$, $\pi(y)=\bar{y}$ and $\pi: u^\perp\mapsto\bar{u}^\perp$ is the canonical projection. Also if $u\in T_pM$ is null, define the Jacobi operator $\bar{R}_u:\bar{u}^\perp\mapsto\bar{u}^\perp$ by $\bar{R}_u\bar{x}=\pi(R(x,u)u)$, where $x\in u^\perp$ with $\pi(x)=\bar{x}$ and R is the curvature tensor.

The Einstein tensor G of a semi-Riemannian manifold is defined by $G=\mathrm{Ric}-\frac{1}{2}(\mathrm{Sc})\langle\ ,\ \rangle$, where Ric and (Sc) are the Ricci tensor and the scalar curvature, respectively. Also note that the divergence $\operatorname{div} G=0$ (see ref. [11, p. 336]). The Ricci operator $\mathfrak{R}: TM\to TM$ is defined by $\langle\mathfrak{R}x, y\rangle=\mathrm{Ric}(x,y)$ for every x, $y\in T_pM$ at each $p\in M$.

3. Null isotropy

An observer in cosmological circumstances observes that the density of light is (ideally) locally uniform on his celestial sphere (null isotropy) rather than uniform over all his celestial sphere (for infinitesimal null isotropy see refs. [5,9]). We shall formulate this concept as follows:

Definition 3.1. Let M be an indefinite semi-Riemannian manifold of $\dim M\geqslant 3$. A null $u\in T_pM$ is called isotropic if $\bar{R}_u=c_u\overline{\mathrm{Id}}$, where $c_u\in\mathbb{R}$. M is called null isotropic at $p\in M$ if every null $u\in T_pM$ is isotropic. M is called null isotropic if M is null isotropic at each $p\in M$.

Remark 3.2.

(1) Every three-dimensional indefinite semi-Riemannian manifold is null isotropic, since $\dim(\bar{u}^{\perp})=1$ for every null $u\in TM$.

(2) If $\bar{R}_u=c_u\overline{\mathrm{Id}}$, then $c_u=(n-2)^{-1}\mathrm{Ric}(u,u)$.

(3) If $\bar{R}_u=c_u\overline{\mathrm{Id}}$, then

$$c_u=\langle R(x,u)u,x\rangle/\langle x,x\rangle$$

for every nonnull $x\in u^{\perp}$. Conversely, if $\langle x,x\rangle c_u=\langle R(x,u)u,x\rangle$ for every spacelike (or timelike) $x\in u^{\perp}$ then $\bar{R}_u=c_u\overline{\mathrm{Id}}$.

(4) If the index $2\leqslant \nu\leqslant n-2$, then $\bar{R}_u=c_u\overline{\mathrm{Id}}$ iff $\langle R(v,u)u,v\rangle=0$ for every null $v\in u^{\perp}$.

For the proofs of (3) and (4), apply lemmas A and B of ref. [10] to $f(\bar{x},\bar{y})=\bar{g}(\bar{R}_u\bar{x},\bar{y})$ defined on $\bar{u}^{\perp}$.

Next we will obtain the curvature tensor of a null isotropic indefinite semi-Riemannian manifold which will be a generalization of the curvature tensor of constant curvature semi-Riemannian manifolds. But for this, we need ref. [3, thm. 1a], in a more general setting.

Let M be a semi-Riemannian manifold. A quadrilinear function $G:T_pM\times T_pM\times T_pM\times T_pM\to\mathbb{R}$ is called curvature-like if:

(a) $G(x,y,z,v)=-G(y,x,z,v)=-G(x,y,v,z)$,

(b) $G(x,y,z,v)+G(y,z,x,v)+G(z,x,y,v)=0$,

(c) $G(x,y,z,v)=G(z,v,x,y)$,

for every $x, y, z, v\in T_pM$. Also, let G be a curvature-like quadrilinear function on T_pM. Then for $x, y, z\in T_pM$, there is a unique element in T_pM, which we denote by $R'(x,y)z$, such that $\langle R'(x,y)z,v\rangle=G(x,y,z,v)$ for every $v\in T_pM$. The map $R':T_pM\times T_pM\times T_pM\to T_pM$ defined by the above equation is called the curvature tensor of G.

Clearly, the curvature tensor of a curvature-like quadrilinear function is trilinear and satisfies

(a′) $R'(x,y)z=-R'(y,x)z$,

(b′) $R'(x,y)z+R'(y,z)x+R'(z,x)y=0$,

for every $x, y, z\in T_pM$. Also let F and F_0 be the quadrilinear curvature-like functions on T_pM defined by $F(x,y,z,v)=\langle R(x,y)z,v\rangle$ and $F_0(x,y,z,v)=\langle R_0(x,y)z,v\rangle$, where

$$R_0(x,y)z=\langle y,z\rangle x-\langle x,z\rangle y\,. \tag{1}$$

Note that the proof of theorem 1a in ref. [3] only depends on the properties of a curvature-like quadrilinear function and its curvature tensor. Hence we can also state this theorem in a more general way as follows.

Theorem 3.3. *Let M be a semi-Riemannian manifold and let G be a curvature-like quadrilinear function on* T_pM. *If* $G(x, y, z, x) = \langle R'(x, y)z, x\rangle = 0$ *whenever* $\{x, y\}$ *is orthonormal of signature* $(-, +)$ *(or* $(+, +)$ *or* $(-, -)$*) and* $\langle x, z\rangle = \langle y, z\rangle = 0$, *then* $G = CF_0$, *where* $C \in \mathbb{R}$.

Let M be a semi-Riemannian manifold of $\dim M \geqslant 3$ and let $R_1 : T_pM \times T_pM \times T_pM \to T_pM$ be a trilinear map defined by

$$R_1(x, y)z = \mathrm{Ric}(z, y)x - \mathrm{Ric}(z, x)y$$

at each $p \in M$. Then it is easy to see that the function $F'_0 : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ defined by

$$F'_0(x, y, z, v) = \frac{1}{n-2}\left[\langle R_1(x, y)z, v\rangle + \mathrm{Ric}(R_0(x, y)z, v)\right] \tag{2}$$

is a curvature-like quadrilinear function on T_pM.

Also note that

$$F'_0(x, u, u, x) = \frac{1}{n-2}\mathrm{Ric}(u, u)\langle x, x\rangle$$

for every null $u \in T_pM$ and $x \in u^\perp$.

Remark 3.4. If M is null isotropic at $p \in M$ then

$$F'_0(x, u, u, x) = c_u\langle x, x\rangle$$

for every null $u \in T_pM$ and $x \in u^\perp$, where $\bar{R}_u = c_u\overline{\mathrm{Id}}$.

Theorem 3.5. *Let M be an indefinite semi-Riemannian manifold. If M is null isotropic at* $p \in M$, *then*

$$F = F'_0 - \frac{1}{(n-1)(n-2)}(\mathrm{Sc})_p F_0\,. \tag{3}$$

Proof. Let $G = F - F'_0$ be a curvature-like quadrilinear function on T_pM. Then $G(x, u, u, x) = 0$ for every null $u \in T_pM$ and $x \in u^\perp$ by remarks 3.2(3) and 3.4. Let $x \in T_pM$ be a unit nonnull vector with $x^\perp$ indefinite and let $f(y, z) = G(x, y, z, x)$ be a symmetric bilinear function on $x^\perp$. Then $f(v, v) = 0$ for every null $v \in x^\perp$ and it follows from ref. [10, lemma A] that $f(y, z) = \lambda_x\langle y, z\rangle$ on $x^\perp$. Thus $G(x, y, z, x) = \lambda_x\langle y, z\rangle$ on $x^\perp$ and it follows that $G(x, y, z, x) = 0$ for every orthogonal $y, z \in x^\perp$. Hence it follows from theorem 3.1 that $G = CF_0$, that is, $F = F'_0 + CF_0$. Now it remains to determine the constant C. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis at $p \in M$. Then

$$\sum_{i,j=1}^{n} \langle e_i, e_i\rangle\langle e_j, e_j\rangle F(e_i, e_j, e_j, e_i)$$
$$= \sum_{i,j=1}^{n} \langle e_i, e_i\rangle\langle e_j, e_j\rangle F'_0(e_i, e_j, e_j, e_i)$$
$$+ C \sum_{i,j=1}^{n} \langle e_i, e_i\rangle\langle e_j, e_j\rangle F_0(e_i, e_j, e_j, e_i) .$$

Hence an easy computation shows that

$$(\mathrm{Sc})_p = \frac{2(n-1)}{n-2}(\mathrm{Sc})_p + Cn(n-1)$$

and it follows that

$$C = -\frac{(\mathrm{Sc})_p}{(n-\cancel{1)(n}-2)} . \qquad \square$$

Recall that the Weyl tensor of a semi-Riemannian manifold of dim $M \geqslant 3$ is defined to be the curvature tensor W of

$$W = F - F'_0 + \frac{1}{(n-1)(n-2)}(\mathrm{Sc})F_0 .$$

Hence we have the following:

Corollary 3.6. *Let M be an indefinite semi-Riemannian manifold of* dim $M \geqslant 3$. *Then M is null isotropic at $p \in M$ iff $W = 0$ at $p \in M$.*

Proof. Obvious. □

Notice that, by the above corollary, the Weyl tensor of an indefinite semi-Riemannian manifold M may also be considered as a measure of the deviation from M being null isotropic.

Corollary 3.7. *Let M be an indefinite semi-Riemannian manifold. If M is null isotropic at $p \in M$, then M is of constant curvature at p iff M is Einstein at p.*

Proof. Obvious. (Also see ref. [10, lemma A and lemma B].) □

Most interesting examples of null isotropic Lorentzian manifolds are the Robertson–Walker spacetimes [11, p. 341]. In the proposition below, we will also construct other examples of null isotropic indefinite semi-Riemannian manifolds. Throughout this paper, let M_1 and M_2 be semi-Riemannian manifolds with metrics $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, curvature tensors R_1 and R_2, respectively. Also let $M=M_1\times M_2$ with metric $\langle\ ,\ \rangle=-\langle\ ,\ \rangle_1\oplus\langle\ ,\ \rangle_2$ (or $\langle\ ,\ \rangle=\langle\ ,\ \rangle_1\oplus-\langle\ ,\ \rangle_2$) and curvature tensor R.

Proposition 3.8. *Let M_1 and M_2 be semi-Riemannian manifolds with $M=M_1\times M_2$ an indefinite semi-Riemannian manifold and* $\dim M\geqslant 3$. *Then M is null isotropic if either of the following holds:*

(1) $\dim M_i\geqslant 2$ *and M_i is of constant curvature c for $i=1,2$.*

(2) $\dim M_1=1$, $\dim M_2\geqslant 3$ *and M_2 is of constant curvature c (or vice versa).*

(3) $\dim M=3$.

Proof. Let $u=(u_1,u_2)\in T_pM$ be a null vector and let $x=(x_1,x_2)\in u^{\perp}$. Hence note that $\langle u_1,u_1\rangle_1=\langle u_2,u_2\rangle_2$ and $\langle x_1,u_1\rangle_1=\langle x_2,u_2\rangle_2$.

(1) Thus

$$\begin{aligned}R(x,u)u&=(R_1(x_1,u_1)u_1,R_2(x_2,u_2)u_2)\\&=c(\langle u_1,u_1\rangle_1x_1-\langle x_1,u_1\rangle_1u_1,\\&\qquad\langle u_1,u_1\rangle_1x_2-\langle x_1,u_1\rangle_1u_2)\\&=c\langle u_1,u_1\rangle_1(x_1,x_2)-c\langle x_1,u_1\rangle_1(u_1,u_2)\,,\end{aligned}$$

and it follows that $\bar{R}_u\bar{x}=c\langle u_1,u_1\rangle_1\bar{x}=c_u\bar{x}$ for every null $u\in T_pM$ and $\bar{x}\in\bar{u}^{\perp}$, where $\pi(x)=\bar{x}$.

(2) Since $\dim M_1=1$, $x_1=ku_1$ and hence $\langle x_1,u_1\rangle_1=k\langle u_1,u_1\rangle_1$. Thus

$$\begin{aligned}R(x,u)u&=(0,R_2(x_2,u_2)u_2)\\&=(0,c(\langle u_1,u_1\rangle_1x_2-k\langle u_1,u_1\rangle_1u_2))\\&=c\langle u_1,u_1\rangle_1(0,x_2-ku_2)\\&=c\langle u_1,u_1\rangle_1(x_1-ku_1,x_2-ku_2)\\&=c\langle u_1,u_1\rangle_1(x_1,x_2)-kc\langle u_1,u_1\rangle_1(u_1,u_2)\,,\end{aligned}$$

and it follows that

$$\bar{R}_u\bar{x}=c\langle u_1,u_1\rangle_1\bar{x}=c_u\bar{x}$$

for every null $u\in T_pM$ and $\bar{x}\in\bar{u}^{\perp}$.

(3) Immediate by remark 3.2(1). □

We also have the following converse to the above proposition:

Theorem 3.9. *Let $M = M_1 \times M_2$ be a product of two semi-Riemannian manifolds and assume that M is null isotropic.*

(1) If $\dim M_i \geqslant 2$ *then M_i is of constant curvature c for $i = 1,2$.*

(2) If $\dim M_1 = 1$ *and* $\dim M_2 \geqslant 3$ *with M_2 is connected, then M_2 is of constant curvature c.*

Proof. Let $p = (p_1, p_2) \in M$, $u = (u_1, u_2) \in T_pM$ be a null vector and $x = (x_1, x_2) \in u^\perp$. Then, since

$$R(x, u)u = (R_1(x_1, u_1)u_1, R_2(x_2, u_2)u_2)$$
$$= c_u(x_1, x_2) + d_x(u_1, u_2)$$
$$= (c_u x_1 + d_x u_1, c_u x_2 + d_x u_2) ,$$

it follows that

$$R_1(x_1, u_1)u_1 = c_u x_1 + d_x u_1 ,$$
$$R_2(x_2, u_2)u_2 = c_u x_2 + d_x u_2 .$$

(1) Note that if $\dim M_i = 2$ then M_i has constant curvature c_{p_i} at each $p_i \in M_i$. Assume $\dim M_i \geqslant 3$. Then, since $0 = \langle R_i(x_i, u_i)u_i, z_i \rangle = \langle R_i(u_i, x_i)z_i, u_i \rangle$ for every orthonormal x_i, z_i, $u_i \in T_{p_i}M_i$ described in theorem 3.9, it follows that M_i is of constant curvature c_{p_i} at $p_i \in M_i$, where $i = 1, 2$. Furthermore let $u = (u_1, u_2) \in T_pM$ be a null vector with $\langle u_1, u_1 \rangle_1 \neq 0$ and $x = (x_1, x_2) \in u^\perp$ be a nonnull vector with x_1 and x_2 not scalar multiples of u_1 and u_2, respectively. Then from

$$R(x, u)u = (R_1(x_1, u_1)u_1, R_2(x_2, u_2)u_2)$$
$$= c_u(x_1, x_2) + d_x(u_1, u_2)$$

it follows that

$$c_{p_1} \langle u_1, u_1 \rangle_1 x_1 - c_{p_1} \langle x_1, u_1 \rangle_1 u_1 = c_u x_1 + d_x u_1 ,$$
$$c_{p_2} \langle u_1, u_1 \rangle_1 x_2 - c_{p_2} \langle x_1, u_1 \rangle_1 u_2 = c_u x_2 + d_x u_2 .$$

Thus $c_{p_1} = c_{p_2}$ for every $p = (p_1, p_2) \in M$ and it follows that M_1 and M_2 are of constant curvature c.

(2) As in (1), M_2 is of constant curvature at each $p_2 \in M_2$. Then, since M_2 is connected it follows from Schur's lemma that M_2 is of constant curvature c. □

Next we will obtain a local splitting theorem for null isotropic semi-Riemannian manifolds. We shall use the following form of the Primary Decomposition Theorem in linear algebra for (not necessarily definite) inner-product spaces.

Theorem 3.10. *Let $\mathfrak{R}$ be a self-adjoint operator on a real inner-product space V. Let m be the minimal polynomial for $\mathfrak{R}$, $m = m_i^{r_1} \cdots m_k^{r_k}$, where m_i are distinct irre-*

ducible nonconstant monic polynomials and r_i are positive integers. Also let $W_i=\ker(m_i(\mathfrak{R})^{r_i})$. Then

(1) W_i, $i=1, ..., k$ are mutually orthogonal (hence nondegenerate) and $V=W_1\oplus\cdots\oplus W_k$.

(2) Each W_i is invariant under $\mathfrak{R}$.

(3) If $\mathfrak{R}_i$ is the operator induced on W_i by $\mathfrak{R}$ then the minimal polynomial for $\mathfrak{R}_i$ is $m_i^{r_i}$.

Proof. See ref. [7, p. 220]. The proof of (1) can be obtained from the primary decomposition by using the fact that $\mathfrak{R}$ is self-adjoint. (Also see ref. [11, p. 260].) □

In the above theorem, $\mathfrak{R}$ is said to have nontrivial primary decomposition if $i\geqslant 2$.

Theorem 3.11. *Let M be a connected, null isotropic indefinite semi-Riemannian manifold with parallel Ricci tensor. If the Ricci operator has nontrivial primary decomposition at $p\in M$ then M is locally a product $M=M_1\times M_2$ of semi-Riemannian manifolds such that*

(1) M_i is of constant curvature c provided that dim $M_i\geqslant 2$ *for* $i=1, 2$;

(2) M_2 is of constant curvature provided that dim $M_1=1$ *(or vice versa).*

Proof. Let $\mathfrak{R}$ have a nontrivial primary decomposition at $p\in M$. Then, since $\mathfrak{R}$ is parallel, the minimal polynomial m of $\mathfrak{R}$ has constant coefficients on M, and hence $m=m_1^{r_1}\cdots m_k^{r_k}$ with the same factors $m_1, ..., m_k$ at each $p\in M$. Then $W_1=\ker(m_1(\mathfrak{R})^{r_1})$ and $V_1=\ker(m_2(\mathfrak{R})^{r_2}\cdots m_k(\mathfrak{R})^{r_1})$ are orthogonal nondegenerate distributions on M with complementary dimensions. Also, since $\mathfrak{R}$ is parallel, if $X, Y\in\Gamma W_1$ then $\nabla_X Y\in\Gamma W_1$ because $(m_1(\mathfrak{R})^{r_1})(\nabla_X Y)=\nabla_X(m_1(\mathfrak{R})^{r_1})(Y)=0$. Similarly, if $X, Y\in\Gamma V_1$, then $\nabla_X Y\in\Gamma V_1$. Thus W_1 and V_1 are integrable with totally geodesic integral manifolds. Thus locally $M=M_1\times M_2$, where M_1 and M_2 are semi-Riemannian manifolds.

Furthermore, since M is null isotropic, by theorem 3.9, M_1 and M_2 are of constant curvature c provided that dim $M_i\geqslant 2$. Also from the same theorem, it follows that M_2 is of constant curvature c, provided that dim $M_1=1$ and dim $M_2\geqslant 3$ (or vice versa). Hence it remains to show that M_2 is of constant curvature c if dim $M_1=1$ and dim $M_2=2$ (or vice versa). But that easily follows from the fact that the Ricci tensor is parallel and $\mathrm{Ric}=0\langle\ ,\ \rangle_1\oplus c_{p2}\langle\ ,\ \rangle_2$, where $\langle\ ,\ \rangle_2$ is the induced metric on M_2. □

Here we note that the Einstein static universe is an example to the theorem above (cf. ref. [1, p. 130]).

Remark 3.12. Note that, by the assumption of the above theorem, the minimal polynomial of $\mathfrak{R}$ must necessarily be $m_1=(t-\rho)(t-\lambda)$, where $\rho, \lambda \in \mathbb{R}$ with $\rho \neq \lambda$. This can be easily seen from the Ricci tensor

$$\mathrm{Ric}=c(n_1-1)\langle\ ,\ \rangle_1 \oplus c(n_2-1)\langle\ ,\ \rangle_2,$$

where $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are the induced metrics on M_1 and M_2 respectively, and $\dim M_1=n_1$ and $\dim M_2=n_2$. Hence $\rho=c(n_1-1)$ and $\lambda=c(n_2-1)$.

4. Infinitesimal isotropy

Definition 4.1. *Let M be a semi-Riemannian manifold. M is called infinitesimally isotropic with respect to orthogonal decomposition $T_pM=W_1 \oplus W_2$ at $p \in M$ if:*

(a) $R(z,x)y=\mu\langle x,y\rangle z$, for every $z \in W_1$, $x, y \in W_2$ and vice versa, where $\mu \in \mathbb{R}$;

(b) $R(x,y)z=\kappa_i R_0(x,y)z$, for every $x, y, z \in W_i$ for $i=1,2$ where $\kappa_i \in \mathbb{R}$. In particular, define $\kappa_i=0$ if $\dim W_i=1$.

Remark 4.2. Note that the above definition is equivalent to that of Karcher [8] if $\dim W_1=1$ (see also ref. [4]). Also condition (a) implies that $R(x,y)z=0$ for every $z \in W_1$, $x, y \in W_2$ and vice versa by the first Bianchi identity.

Examples

1. The Robertson–Walker spacetimes are infinitesimally isotropic with respect to $T_pM=W_1 \oplus W_2$, where $W_1=\mathrm{span}\{Z_p\}$, $W_2=W_1^{\perp}$ and Z is the fluid flow (see ref. [11, p. 345]).

2. Let M_1 and M_2 be semi-Riemannian manifolds with constant curvature c_1 and c_2, respectively. Then $M=M_1 \times M_2$ is infinitesimally isotropic with respect to $T_pM=T_{p_1}M_1 \times T_{p_2}M_2$ at each $p=(p_1,p_2) \in M$. Hence note that infinitesimal isotropy does not imply null isotropy in general (see theorem 3.9).

3. The Kruskal black hole is infinitesimally isotropic with respect to $T_pM=T_{p_1}Q \times T_{p_2}S^2$, where Q is the Kruskal plane and $p=(p_1,p_2) \in Q \times S^2$ (see ref. [11, p. 369]). Also note that the Kruskal black hole is not null isotropic.

Proposition 4.3. *Let M be an indefinite semi-Riemannian manifold and $T_pM=W_1 \oplus W_2$, where W_1 and W_2 are orthogonal. Let $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ be restrictions of $\langle\ ,\ \rangle$ on W_1 and W_2. If M is null isotropic at $p \in M$ and $\mathrm{Ric}=\rho\langle\ ,\ \rangle_1 \oplus \lambda\langle\ ,\ \rangle_2$ then M is infinitesimally isotropic with respect to $T_pM=W_1 \oplus W_2$, where $\rho, \lambda \in \mathbb{R}$.*

Proof. With a straightforward computation using theorem 3.5, it can be shown that M is infinitesimally isotropic with respect to $T_pM=W_1 \oplus W_2$ with

$$\mu = -\frac{1}{n-2}\left(\frac{1}{n-1}(\mathrm{Sc})_p + \rho - \lambda\right),$$

$$\kappa_1 = -\frac{1}{n-2}\left(\frac{1}{n-1}(\mathrm{Sc})_p + \rho\right),$$

$$\kappa_2 = -\frac{1}{n-2}\left(\frac{1}{n-1}(\mathrm{Sc})_p - \lambda\right). \qquad \square$$

The converse of the above proposition is not true in general, see the above example 2 with $c_1 \neq c_2$. But we have the following partial converse.

Proposition 4.4. *If M is an infinitesimal isotropic semi-Riemannian manifold with respect to $T_pM = W_1 \oplus W_2$ at $p \in M$ then*

$$\mathrm{Ric} = -[\kappa_1(n_1 - 1) + \mu n_2]\langle\ ,\ \rangle_1 \oplus [\kappa_2(n_2 - 1) + \mu n_1]\langle\ ,\ \rangle_2, \tag{4}$$

where $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are the restrictions of $\langle\ ,\ \rangle$ on W_1 and W_2, respectively, and $\dim W_1 = n_1$, $\dim W_2 = n_2$.

Proof. Let $e_1, \ldots, e_{n_1} \in W_1$ and $e_{n_1+1}, \ldots, e_{n_1+n_2} \in W_2$ be an orthonormal basis for W_1 and W_2, respectively. Then for $z \in W_1$ and $x \in W_2$,

$$\mathrm{Ric}(z, x) = \sum_{i=1}^{n_1} \langle e_i, e_i\rangle \langle R(e_i, z)x, e_i\rangle + \sum_{i=n_1+1}^{n_1+n_2} \langle e_i, e_i\rangle \langle R(e_i, z)x, e_i\rangle = 0$$

by curvature identities. Also, for $z, v \in W_1$,

$$\mathrm{Ric}(z, v) = \sum_{i=1}^{n_1} \langle e_i, e_i\rangle \langle R(e_i, z)v, e_i\rangle + \sum_{i=n_1+1}^{n_1+n_2} \langle e_i, e_i\rangle \langle R(e_i, z)v, e_i\rangle = [\kappa_1(n_1 - 1) + n_2\mu]\langle z, v\rangle,$$

and similarly, for $x, y \in W_2$,

$$\mathrm{Ric}(x, y) = [\kappa_2(n_2 - 1) + \mu n_1]\langle x, y\rangle.$$

Hence

$$\mathrm{Ric} = -[\kappa_1(n_1-1)+\mu n_2]\langle\ ,\ \rangle_1 \oplus [\kappa_2(n_2-1)+\mu n_1]\langle\ ,\ \rangle_2 . \qquad \square$$

Corollary 4.5. *Let M be a semi-Riemannian manifold. If M is null isotropic at p and infinitesimally isotropic with respect to $T_pM = W_1 \oplus W_2$, then M is of constant curvature at p provided that*

$$\mu(n_2-n_1)+\kappa_1(n_1-1)-\kappa_2(n_2-1)=0 . \tag{5}$$

Proof. Note that, if the above equation is satisfied, then M is Einstein at $p\in M$ by proposition 4.4. Hence M is of constant curvature by corollary 3.7. $\square$

Remark 4.6. Note that $\mu(n_2-n_1)+\kappa_1(n_1-1)-\kappa_2(n_2-1)=0$ does not imply that M is of constant curvature unless M is null isotropic at p. For example, the Kruskal black hole satisfies the above identity but is not of constant curvature. In fact, in the Kruskal black hole, $\kappa_1=\kappa_2=-2\mu\neq 0$.

Next we will show that, if M is infinitesimally isotropic with respect to $T_pM = W_1 \oplus W_2$ with $\dim(W_1)=1$, then we have the following converse of proposition 4.7.

Theorem 4.7. *Let M be an indefinite semi-Riemannian manifold and $T_pM = W_1 \oplus W_2$, where W_1 and W_2 are orthogonal with* $\dim W_1 = 1$ *(or* $\dim W_2 = 1$*). Then M is infinitesimally isotropic with respect to $T_pM = W_1 \oplus W_2$ iff M is null isotropic at p and* $\mathrm{Ric} = \rho\langle\ ,\ \rangle_1 \oplus \lambda\langle\ ,\ \rangle_2$, *where $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are the restrictions of $\langle\ ,\ \rangle$ on W_1 and W_2, respectively.*

Proof. The "if" part follows by proposition 4.3. Also by proposition 4.4 and corollary 3.6, it suffices to show that the Weyl tensor $W=0$ to prove the "only if" part. For it is straightforward to show from

$$W = F - F_0' + \frac{1}{(n-1)(n-2)}(\mathrm{Sc})_p F_0$$

that $W(x,y)v = W(x,z)z = W(x,y)z = W(z,x)y = 0$ for every $z\in W_1$ and $x, y, v\in W_2$. Hence, since any $v, w\in T_pM$, there exist $a, b\in\mathbb{R}$ and $x, y\in W_2$, $z\in W_1$ such that $v=az+x$, $w=bz+y$, it can be shown that $W(v,w)w = W(az+x, bz+y)(bz+y)=0$. Thus, $W(v,w,w,v)=0$ for every $v, w\in T_pM$ and since W is curvature-like, it follows that $W=0$. $\square$

In a spacetime M, null isotropy is independent of a particular observer contrary to infinitesimal null isotropy. (Note that infinitesimal null isotropy is not defined in non-Lorentzian manifolds, see refs. [5,9].) Indeed, from a physical viewpoint, M may be seen to be infinitesimally null isotropic by one observer yet may not be seen that way by another observer. (Actually, it can be seen from the uniqueness of the eigenspaces of the Ricci operator that, if M is infinitesimally null isotropic (equivalently, infinitesimally spatially isotropic) with respect to two linearly independent observers then M must be of constant curvature at that point.) Hence it may be interesting to give a necessary and sufficient condition for the existence of an observer in a null isotropic spacetime for whom M is infinitesimally null isotropic. This condition is clearly that the stress–energy tensor T of M be that of a perfect fluid, as is immediate from the Einstein equation, theorem 4.7 and ref. [9, thm. 1]. In this case, the observer is the unit future-directed time-like eigenvector of T.

Let M be a semi-Riemannian manifold and let $TM = W_1 \oplus W_2$, where W_1 and W_2 are smooth orthogonal distributions. M is called infinitesimally isotropic with respect to $TM = W_1 \oplus W_2$ if M is infinitesimally isotropic with respect to $T_pM = W_{1_p} \oplus W_{2_p}$ at each $p \in M$.

Note that, in this case, μ, κ_1, $\kappa_2 : M \to \mathbb{R}$ are smooth functions.

Theorem 4.8. *Let M be a semi-Riemannian manifold. Assume that M is infinitesimally isotropic with respect to $TM = W_1 \oplus W_2$ with* $\operatorname{rank}(W_2) \geq 3$. *If*

(a) $\kappa_2 - \mu \neq 0$ at each $p \in M$,

(b) $d\mu|_{W_2} = 0$,

then M is locally a warped product $(M_1 \times_f M_2, -\langle\ ,\ \rangle_1 \oplus f\langle\ ,\ \rangle_2)$, where M_1 and M_2 are semi-Riemannian manifolds with M_2 of constant curvature, M_1 of constant curvature provided that $\dim M_1 \geq 3$, *and $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are the induced metrics on W_1 and W_2, respectively.*

Proof. Let $Z \in \Gamma W_1$ and $X, Y, V \in \Gamma W_2$. Then

$$(\nabla_Z R)(X, Y)V = d\kappa_2(Z) R_0(X, Y)V$$
$$+ (\mu - \kappa_2)[\langle X, V\rangle (\nabla_Z Y)^{T_1} - \langle V, Y\rangle (\nabla_Z X)^{T_1}],$$
$$(\nabla_X R)(Y, Z)V = -d\mu(X)\langle Y, V\rangle Z$$
$$+ (\kappa_2 - \mu)[\langle Y, V\rangle (\nabla_X Z)^{T_2} - \langle V, \nabla_X Z\rangle Y],$$
$$(\nabla_Y R)(Z, X)V = d\mu(Y)\langle X, V\rangle Z$$
$$- (\kappa_2 - \mu)[\langle X, V\rangle (\nabla_Y Z)^{T_2} - \langle V, \nabla_Y Z\rangle X],$$

where T_1 and T_2 are the projections on W_1 and W_2, respectively. Then by the second Bianchi identity, we obtain that

$$[d\mu(Y)\langle X, V\rangle - d\mu(X)\langle Y, V\rangle]Z$$
$$= (\kappa_2 - \mu)[\langle X, V\rangle(\nabla_Z Y)^{T_1} - \langle V, Y\rangle(\nabla_Z X)^{T_1}], \quad (6)$$
$$d\kappa_2(Z)R_0(X, Y)V = (\kappa_2 - \mu)[\langle V, \nabla_X Z\rangle Y$$
$$- \langle Y, V\rangle(\nabla_X Z)^{T_2} + \langle X, V\rangle(\nabla_Y Z)^{T_2} - \langle V, \nabla_Y Z\rangle X]. \quad (7)$$

By assumption, since $d\mu|_{W_2} = 0$ and $\kappa_2 - \mu \neq 0$ at each $p \in M$, we obtain from (6) that

$$\langle X, V\rangle(\nabla_Z Y)^{T_1} - \langle V, Y\rangle(\nabla_Z X)^{T_1} = 0$$

for every $Z \in \Gamma W_1$ and $X, Y, V \in \Gamma W_2$. Thus by taking $Y = V \perp X$ and multiplying the above equation with $Z' \in \Gamma W_1$, we obtain

$$0 = -\langle Y, Y\rangle\langle \nabla_Z X, Z'\rangle = \langle Y, Y\rangle\langle X, \nabla_Z Z'\rangle$$

for every $X, Y \in \Gamma W_2$ and $Z, Z' \in \Gamma W_1$.

Thus W_1 is integrable and integral manifolds are totally geodesic. Hence these integral manifolds are locally of constant curvature by Schur's lemma if their dimension is $\geqslant 3$. Also from (7), by choosing $Y = V \perp X$ and then multiplying it with X, we obtain

$$\frac{d\kappa_2(Z)}{\mu - \kappa_2} = \frac{\langle \nabla_X Z, X\rangle}{\langle X, X\rangle} + \frac{\langle \nabla_Y Z, Y\rangle}{\langle Y, Y\rangle}$$

for every orthogonal nonnull $X, Y \in \Gamma W_2$. But since $\operatorname{rank}(W_2) \geqslant 3$, we conclude that

$$\frac{\langle \nabla_X Z, X\rangle}{\langle X, X\rangle} = \frac{1}{2}\frac{d\kappa_2(Z)}{\mu - \kappa_2}$$

for every nonnull $X \in \Gamma W_2$, and hence

$$\langle \nabla_X Z, X\rangle = \frac{1}{2}\frac{d\kappa_2(Z)}{\mu - \kappa_2}\langle X, X\rangle \quad (8)$$

for every $X \in \Gamma W_2$.

Also from (7) by choosing orthogonal $X, Y, V \in \Gamma W_2$, with X, Y linearly independent, we obtain $\langle V, \nabla_X Z\rangle Y - \langle V, \nabla_Y Z\rangle X = 0$, and hence

$$\langle V, \nabla_X Z\rangle = 0 \quad (9)$$

for every nonnull orthogonal $X, V \in \Gamma W_2$ and $Z \in \Gamma W_1$.

Now let $Z \in \Gamma W_1$ and define a bundle homomorphism $T: W_2 \to W_2$ by

$$T = (\nabla Z)^{T_2} - \frac{1}{2}\frac{d\kappa_2(Z)}{\mu - \kappa_2}\,\mathrm{Id}.$$

Then by (8),

$$\langle TX, X\rangle = \langle \nabla_X Z, X\rangle - \frac{1}{2}\frac{d\kappa_2(Z)}{\mu-\kappa_2}\langle X, X\rangle = 0$$

for every $X \in \Gamma W_2$, and by (9), $\langle TX, Y\rangle = 0$ for every nonnull orthogonal $X, Y \in \Gamma W_2$. Thus $T=0$ and hence

$$(\nabla Z)^{T_2} = \frac{1}{2}\frac{d\kappa_2(Z)}{\mu-\kappa_2}\,\mathrm{Id}\,.$$

But this implies that W_2 is integrable and the integral manifolds are totally umbilic. (Note that the second fundamental form operator $L_Z = -(\nabla Z)^{T_2}$ with respect to $Z \in \Gamma W_1$ is scalar.) Hence the integral manifolds of W_2 have constant curvature in the induced metric at each point and it follows from Schur's lemma that these integral manifolds are locally of constant curvature. □

The above theorem is a generalization of a theorem of Karcher [8], and in particular it characterizes cosmological metrics in General Relativity. For its physical interpretation in cosmological circumstances, see refs. [8,5,9].

However, the above theorem fails to hold in black hole models since rank $W_2 \geqslant 3$ is assumed. Next we shall state another version of this theorem which holds in static black hole models and characterizes the static black hole metrics.

Theorem 4.9. *Let M be a semi-Riemannian manifold. Assume that M is infinitesimally isotropic with respect to $TM = W_1 \oplus W_2$ with* $\mathrm{rank}(W_1) \geqslant 2$ *and* $\mathrm{rank}(W_2) = 2$. *If*

(a) $\kappa_1 - \mu \neq 0$ and $\kappa_2 - \mu \neq 0$ at each $p \in M$,

(b) $d\mu|_{W_2} = 0$,

(c) $d\kappa_2|_{W_2} = 0$,

then M is locally a warped product $(M_1 \times_f M_2, -\langle\ ,\ \rangle_1 \oplus f\langle\ ,\ \rangle_2)$, where M_1 and M_2 are semi-Riemannian manifolds with M_1 of constant curvature if $\dim M_1 \geqslant 3$, *M_2 of constant curvature, and $-\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are induced metrics on W_1 and W_2, respectively.*

Proof. Let $U, V \in \Gamma W_1$ and $X, Y \in \Gamma W_2$. Then

$$\begin{aligned}(\nabla_X R)(Y,U)V &= d\mu(X)\langle U,V\rangle Y \\ &\quad + (\mu-\kappa_1)[\langle U,V\rangle(\nabla_X Y)^{T_1} - \langle \nabla_X Y, V\rangle U]\,,\\ (\nabla_Y R)(U,X)V &= -d\mu(Y)\langle U,V\rangle X \\ &\quad - (\mu-\kappa_1)[\langle U,V\rangle(\nabla_Y X)^{T_1} - \langle \nabla_Y X, V\rangle U]\,,\\ (\nabla_U R)(X,Y)V &= (\mu-\kappa_2)[\langle Y, \nabla_U V\rangle X - \langle X, \nabla_U V\rangle Y]\,,\end{aligned}$$

where T_1 is the projection on W_1. Then by the second Bianchi identity, we obtain that

$$[d\mu(X)Y-d\mu(Y)X]\langle U,V\rangle$$
$$=(\kappa_2-\mu)[\langle Y,\nabla_U V\rangle X-\langle X,\nabla_U V\rangle Y]\,, \quad (10)$$

$$(\mu-\kappa_1)[\langle U,V\rangle[X,Y]^{T_1}-\langle[X,Y],V\rangle U]=0\,. \quad (11)$$

Also from (10), by interchanging X and Y with U and V, respectively, we obtain

$$[d\mu(U)V-d\mu(V)U]\langle X,Y\rangle$$
$$=(\kappa_1-\mu)[\langle V,\nabla_X Y\rangle U-\langle U,\nabla_X Y\rangle V]\,. \quad (12)$$

By assumption, since $d\mu|_{W_2}=0$ and $\kappa_2-\mu\neq 0$ at each $p\in M$, we obtain

$$\langle Y,\nabla_U V\rangle X-\langle X,\nabla_U V\rangle Y=0$$

for every $U, V\in\Gamma W_1$ and $X, Y\in\Gamma W_2$. Hence $\langle Y,\nabla_U V\rangle=0$ for every $Y\in\Gamma W_2$, U, $V\in\Gamma W_1$, and it follows that W_1 is integrable with totally geodesic integral manifolds. Also from (11), by taking $U\perp V$, we obtain $\langle[X,Y],V\rangle=0$ for every $X, Y\in\Gamma W_2$, $V\in\Gamma W_1$. Thus W_2 is integrable.

Furthermore, from (12)

$$\frac{d\mu(U)}{\kappa_1-\mu}\langle X,Y\rangle=-\langle\nabla_X Y,U\rangle=\langle Y,\nabla_X U\rangle$$

for every $U\in\Gamma W_1$, $X, Y\in\Gamma W_2$. Hence

$$(\nabla U)^{T_2}=\frac{d\mu(U)}{\kappa_1-\mu}\,\mathrm{Id}$$

for every $U\in\Gamma W_1$, which implies that the integral manifolds of W_2 are totally umbilic. Hence it remains to show that these integral manifolds are locally of constant curvature.

For, since κ_2 is locally constant on the integral manifolds of W_2, it follows from ref. [11, p. 124, prob. 6] that these integral manifolds are locally of constant curvature in the induced metric. □

The best examples to the above theorem are Kruskal and Reissner–Nordström black holes [6]. (Yet note that a Reissner–Nordström black hole is not Ricci flat due to the existence of electromagnetic stress–energy.) Perhaps it may also be interesting to give a physical interpretation of infinitesimal isotropy in black hole circumstances. Let W_1 be the Lorentzian space spanned by all observers radial to the black hole at a point p of spacetime M and let $W_2=W_1^{\perp}$. Then an observer $z\in W_1$ has the rest space $z^{\perp}=L\oplus W_2$, where L is the space-like subspace of W_1 orthogonal to span$\{z\}$. Thus the infinitesimal isotropy with respect to $T_pM=W_1\oplus W_2$ implies that the compressing tidal accelerations acting on this observer which are transversal to the black hole (that is, those in W_2) are uniform in every direction and their magnitudes are proportional to μ. Also the pulling effect of the black hole is in the direction of L directed to the black hole (in the rest space of the observer) and its magnitude is proportional to κ_1. κ_2 has a similar meaning in describing the gravity of the black hole in bending the planes transversal to it.

Also see ref. [2] for the description of tidal accelerations in a Kruskal black hole.

Finally we will make a general remark on these results in General Relativity. Due to nonuniform clumpings in the later phases of the universe, the stress–energy tensor may deviate from being a perfect fluid to be a fluid with different principal pressures. Hence it may be an overidealization to expect the universe to be infinitesimally null isotropic (equivalently, infinitesimally spatially isotropic). (Also see the remark below theorem 4.7.) Yet one still expects the universe to be null isotropic at most points of spacetime since stars are not expected to occupy much room in the universe, that is, the regions with significant Weyl tensor. At such points, by the above theorem, one may expect that the universe is locally a warped product in most cases.

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